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Idempotent Ideals in Lie Nilpotent Rings

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Dedicated to the memory of Gábor Révész

1. INTRODUCTION

The aim of the present note is twofold. On one hand we give a generalization of a well known theorem characterizing finitely generated idempotent ideals in commutative rings. On the other hand in proving our result we can illustrate the use of the new theory of determinants presented in [1]. Let R be a ring with 1 and $_RM$ a left R-module, for a subset $S \subseteq R$ let SM denote the subset of M consisting of all (finite) sums of the form $\sum w_i m_i$, with $w_i \in S$ and $m_i \in M$. A (two sided) ideal $\mathcal{A} \lhd R$ is called idempotent if $\mathcal{A}^2 = \mathcal{A}$, where $\mathcal{A}^2$ is defined as SM with $S = M = \mathcal{A}$. For an integer $m \geq 1$ the ring R is called Lie nilpotent of index m (or m-Lie nilpotent) if

$$[[[\cdots[[u_1, u_2], u_3], \ldots], u_m], u_{m+1}] = 0$$

is a PI on R (here $[u, v] = uv - vu$). A ring which is Lie nilpotent of some index m is simply called Lie nilpotent. We shall prove the following:

THEOREM 1. *Let R be a Lie nilpotent ring with the property that $1/k \in R$ for all integers $k \geq 1$ (i.e. let R be a Lie nilpotent algebra over a field*

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of characteristic zero). If $\mathcal{A} \lhd R$ is an idempotent ideal which is finitely generated as a right (or left) ideal, then $\mathcal{A} = eR = Re$ for some idempotent element $e \in \mathcal{A}$.

Before proceeding the proof some comments are in order. On replacing the condition of Lie nilpotency by commutativity and keeping only $1 \in R$ from the second condition, we get the commutative variant of the above theorem, which can be found e.g. in [2, p.154]. We also note that in our proof we shall closely follow the steps of the proof given in [2], the essential difference will lie in the use of the so called right adjoints and right determinants allowing us to handle certain systems of linear equations over Lie nilpotent rings.

A subset $\{x_1, x_2, ..., x_n\} \subseteq M$ in an (R, R)-bimodule ${}_RM_R$ is called a (finite) normalizing generator of M if $x_iR = Rx_i$ for each $1 \le i \le n$ and $Rx_1 + Rx_2 + ... + Rx_n = M$. Clearly, any normalizing generator of M generates M as a left R-module and as a right R-module at the same time (in general, the converse is not true). Since (two sided) ideals in R are (R, R)-bimodules in a natural way, normalizing generators also can be considered for the ideals of R. The following "strong" form of Nakayama's lemma due to Resco ([3]) and Robson-Small ([4]) has certain relevance in our exposition.

THEOREM 2. *Let $S \le R$ be a right ideal and ${}_RM_R$ an (R, R)-bimodule with a finite normalizing generator. If $SM = M$, then we have $(1 - e)M = \{0\}$ for some $e \in S$.*

In particular, if $\mathcal{A}^2 = \mathcal{A}$ then the above theorem and its dual can be applied to $S = M = \mathcal{A}$. Now it is easy to derive that any idempotent ideal $\mathcal{A}$ with a finite normalizing generator is of the form $\mathcal{A} = eR = Re$ for some idempotent element $e \in \mathcal{A}$. We note that this result holds in an arbitrary ring R. Since in general we have $xR \ne Rx$ even in a Lie nilpotent ring R, for an ideal $\mathcal{A} \lhd R$ the existence of a finite normalizing generator is a much stronger condition than being finitely generated as a right (or left) ideal. Thus the above consequence of Theorem 2 doesn't cover Theorem 1.

2. PREREQUISITES AND THE PROOF OF THEOREM 1.

In order to provide a self contained treatment, first we collect some of the basic definitions and results from [1]. Let $A = [a_{ij}] \in M_n(R)$ be an $n \times n$ matrix over an arbitrary ring R. For the permutations $\rho \in \mathrm{Sym}(\{1, 2, ..., n\})$ and $\tau \in \mathrm{Sym}(\{1, ..., s-1, s+1, ..., n\})$ we shall make use of the following product:

$$a(s, \tau, \rho) = a_{\tau(1)\rho(\tau(1))} \cdots a_{\tau(s-1)\rho(\tau(s-1))} a_{\tau(s+1)\rho(\tau(s+1))} \cdots a_{\tau(n)\rho(\tau(n))}.$$

The (two-sided) preadjoint of A is the matrix $A^* = [a^*_{rs}] \in M_n(R)$, where

$$a^*_{rs} = \sum_{(\tau,\rho)} \operatorname{sgn}(\rho) a(s,\tau,\rho)\,, \qquad 1 \le r,s \le n$$

and the sum is taken over all permutations $\tau \in \mathrm{Sym}(\{1,...,s-1,s+1,...,n\})$ and $\rho \in \mathrm{Sym}(\{1,2,...,n\})$ with $\rho(s)=r$. The right adjoint sequence $(A_k)_{k\ge1}$ of A is defined by the following recursion: $A_1 = A^*$ and for $k \ge 1$ let

$$A_{k+1} = (AA_1...A_k)^*\,.$$

For an integer $m \ge 1$, the right m-adjoint of A is defined as $\mathrm{radj}_{(m)}(A) = A_1A_2...A_m$ and the right m-determinant $\mathrm{rdet}_{(m)}(A) \in R$ of A is the $(1,1)$ entry of the product matrix $AA_1...A_m$. We note that another possible definition of $\mathrm{rdet}_{(m)}(A)$ is $(1/n)\mathrm{tr}(AA_1...A_m)$, which coincides with the $(1,1)$ entry of $AA_1...A_m$ for an m-Lie nilpotent R (see [5]). One of the main results in [1] is the following.

THEOREM 2.1. *Let $A \in M_n(R)$, where R is a m-Lie nilpotent ring with* 1. *Then*

$$A\,\mathrm{radj}_{(m)}(A) = \mathrm{rdet}_{(m)}(A)E,$$

where $E \in M_n(R)$ is the unit matrix.

The following two technical lemmas will prove useful.

LEMMA 2.2. *Let $S = \langle a_{ij} \mid 1 \le i,j \le n\rangle$ be a (not necessarily unitary) subring of R generated by the entries of the $n \times n$ matrix $A = [a_{ij}] \in M_n(R)$. For an integer $m \ge 1$ the right m-determinant of the matrix $A - E \in M_n(R)$ is of the form*

$$\mathrm{rdet}_{(m)}(A-E) = (-1)^n[(n-1)!]^{1+n+n^2+...+n^{m-1}}1_R - w,$$

where $w \in S$.

Proof. A slight modification of the proof of Proposition 4.1 in [1] can be used to verify our statement on $\mathrm{rdet}_{(m)}(A-E)$. Let $((A-E)_k)_{k\ge1}$ be the right adjoint sequence of the matrix $A-E$. We claim that in the product matrix $(A-E)(A-E)_1...(A-E)_m$ each non diagonal entry is in S and each diagonal entry is of the form $(-1)^n[(n-1)!]^{1+n+n^2+...+n^{m-1}}1_R - w$ for some $w \in S$. We apply an induction on m. If $m=1$ then a direct computation of the entries of $(A-E)(A-E)^*$ can prove the claim (an argument similar to the following one on TT^* also works). Assume now that our claim holds for $m-1$. Put

$$T = [t_{ij}] = (A-E)(A-E)_1...(A-E)_{m-1}$$

then $(A-E)_m = T^*$ and consider the matrix

$$[b_{ij}] = TT^* = (A-E)(A-E)_1 ... (A-E)_m.$$

It is straightforward to see that

$$\text{(e)} \quad b_{ij} = \sum_{(\tau,\rho)} \operatorname{sgn}(\rho)\, t_{i\rho(j)} t(j,\tau,\rho)$$

where $t(j,\tau,\rho) = t_{\tau(1)\rho(\tau(1))} ... t_{\tau(j-1)\rho(\tau(j-1))} t_{\tau(j+1)\rho(\tau(j+1))} ... t_{\tau(n)\rho(\tau(n))}$ and the sum is taken over the set of all pairs (τ,ρ) with $\tau \in$Sym$(\{1,...,j-1,j+1,...,n\})$ and $\rho \in$Sym$(\{1,2,...,n\})$. Since $kS = Sk \subseteq S$ for all integers $k \in \mathbf{Z}$, the inductive assumption on T implies that any summand in (e) having at least one non-diagonal factor is in S. If $i \neq j$ then it is easy to see that at least one of the terms

$$t_{i\rho(j)}, t_{\tau(1)\rho(\tau(1))}, ..., t_{\tau(j-1)\rho(\tau(j-1))}, t_{\tau(j+1)\rho(\tau(j+1))}, ..., t_{\tau(n)\rho(\tau(n))}$$

is a non diagonal entry of T. In consequence, for $i \neq j$ we get that $b_{ij} \in S$. If $i = j$ then all the summands in (e) having only diagonal factors belong to $\rho =$id. It follows that b_{ii} is a sum of elements from S plus the following expression

$$u_i = \sum_{\tau \in Sym(\{1,...,i-1,i+1,...,n\})} t_{ii} t_{\tau(1)\tau(1)} ... t_{\tau(i-1)\tau(i-1)} t_{\tau(i+1)\tau(i+1)} ... t_{\tau(n)\tau(n)}.$$

For $1 \leq r \leq n$ the assumption on T enables us to write

$$t_{rr} = (-1)^n[(n-1)!]^{1+n+n^2+...+n^{m-2}} 1_R - v_r \text{ with } v_r \in S.$$

On substituting these t_{rr}'s into the above sum, we get that

$$u_i = (n-1)! \left((-1)^n[(n-1)!]^{1+n+n^2+...+n^{m-2}} \right)^n 1_R - \overline{w}_i \text{ for some } \overline{w}_i \in S.$$

Now the earlier observation on b_{ii} gives that

$$b_{ii} = (-1)^n[(n-1)!]^{1+n+n^2+...+n^{m-1}} 1_R - w_i \text{ for some } w_i \in S$$

and this completes the proof.

LEMMA 2.3. *Let R be a Lie nilpotent ring of index m, then $uv = 0$ $(u, v \in R)$ implies that $v^m u = vu^m = 0$.*

Proof. Using the Engelian consequences

$$[[...[[u,v],v],...,v],v] = 0 \text{ and } [[...[[v,u],u],...,u],u] = 0$$

of the m-Lie nilpotency of R, we immediately obtain that $v^m u = vu^m = 0$.

Proof of Theorem 1. Since $\mathcal{A}$ is finitely generated as a right ideal, we have

$$\mathcal{A} = x_1R + x_2R + ... + x_nR$$

for some $x_1, x_2, ..., x_n \in \mathcal{A}$. The idempotent property of $\mathcal{A}$ gives that

$$\mathcal{A} = \mathcal{A}^2 \subseteq x_1R\mathcal{A} + x_2R\mathcal{A} + ... + x_nR\mathcal{A} = x_1\mathcal{A} + x_2\mathcal{A} + ... + x_n\mathcal{A},$$

whence we get the existence of an $n \times n$ matrix $A = [a_{ij}] \in M_n(\mathcal{A})$ such that

$$x_1 = x_1a_{11} + x_2a_{21} + ... + x_na_{n1},$$

$$x_2 = x_1a_{12} + x_2a_{22} + ... + x_na_{n2},$$

$$...$$

$$x_n = x_1a_{1n} + x_2a_{2n} + ... + x_na_{nn}.$$

Clearly, the above system of linear equations can be written in the following matrix form:

$$\mathbf{x}(A - E) = \mathbf{0}$$

with $\mathbf{x} = [x_1, x_2, ..., x_n]$ and $\mathbf{0} = [0, 0, ..., 0]$ being $1 \times n$ matrices. In view of Theorem 2.1, the right multiplication by $\mathrm{radj}_{(m)}(A - E)$ gives that

$$\mathbf{x}(A - E)\mathrm{radj}_{(m)}(A - E) = \mathbf{x}\mathrm{rdet}_{(m)}(A - E) = \mathbf{0},$$

where $m \geq 1$ is the index of the Lie nilpotency of R. By Lemma 2.2, we can write that

$$\mathrm{rdet}_{(m)}(A - E) = k1_R - w$$

with $k = (-1)^n[(n-1)!]^{1+n+n^2+...+n^{m-1}}$ and $w \in \mathcal{A}$. Starting from the equations

$$x_i(k1_R - w) = 0, \quad 1 \leq i \leq n$$

the repeated application of Lemma 2.3 gives first $(k1_R - w)^m\mathcal{A} = \{0\}$ and next $\mathcal{A}(k1_R - w)^{m^2} = \{0\}$. On multiplying the equations

$$(k1_R - w)^{m^2}\mathcal{A} = \mathcal{A}(k1_R - w)^{m^2} = \{0\}$$

by $(1/k)^{m^2}$, we deduce that

$$(1_R - e)\mathcal{A} = \mathcal{A}(1_R - e) = \{0\},$$

where $(1/k)^{m^2}(k1_R - w)^{m^2} = (1_R - (1/k)w)^{m^2} = 1_R - e$ and $e \in \mathcal{A}$. Now $(1_R - e)e = e(1_R - e) = 0$ and $e\mathcal{A} = e\mathcal{A} + (1_R - e)\mathcal{A} = \mathcal{A} = \mathcal{A}e + \mathcal{A}(1_R - e) = \mathcal{A}e$, whence we obtain that $\mathcal{A} = eR = Re$ for the idempotent element $e \in \mathcal{A}$.

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